

Linear projections and successive minima

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Let K be a number field, \mathcal{O}_K its ring of integers and E a projective \mathcal{O}_K -module of finite rank N . We endow $E \otimes_{\mathbb{Z}} \mathbb{C}$ with an hermitian metric h and we let μ_1, \dots, μ_N be the logarithm of the successive minima of (E, h) . Assume $X_K \subset \mathbb{P}(E_K^\vee)$ is a smooth geometrically irreducible curve. In this paper we shall find a lower bound for the numbers μ_i , $3 \leq i \leq N$, in terms of the height of X_K , μ_1 and the average of the μ_i 's (Theorem 2). This result is a complement to [9], Theorem 4, which gives a lower bound for μ_1 .

The method of proof is a variant of [9], *loc. cit.* It relies upon Morrison's proof of the fact that X_K is Chow semi-stable [7]. We use a filtration $V_1 = E_K \supset V_2 \supset \dots \supset V_N$ of the vector space E_K . This filtration is chosen so that, for suitable values of i , the projection $\mathbb{P}(V_i^\vee) \cdots \rightarrow \mathbb{P}(V_{i+1}^\vee)$ does not change the degree of the image of X_K by linear projection. That such a choice is possible follows from a result of C. Voisin, namely an effective version of a theorem of Segre on linear projections of complex projective curves (Theorem 1). I thank her for proving this result and for helpful discussions.

1 Linear projections of projective curves

Let $C \subset \mathbb{P}^n$ be an integral projective curve over \mathbb{C} and d its degree. Assume that C is not contained in some hyperplane, $d \geq 3$ and $n \geq 3$.

Theorem 1. (C. Voisin) *There exists an integer $A = A(d)$ and a finite set Σ of points in $(\mathbb{P}^n - C)(\mathbb{C})$, of order at most A , such that, for every point $P \in \mathbb{P}^n(\mathbb{C}) - \Sigma \cup C(\mathbb{C})$, the linear projection $\mathbb{P}^n \cdots \rightarrow \mathbb{P}^{n-1}$ of center P maps C birationally onto its image.*

Proof. The existence of a finite set Σ with the property above is a special case of a theorem of C. Segre [3]. The order of Σ can be bounded as follows by a function of d .

If $n > 3$ a generic linear projection into \mathbb{P}^3 will map C isomorphically onto its image [6] and the exceptional set $\Sigma \subset \mathbb{P}^n$ bijectively onto the exceptional set in \mathbb{P}^3 . Therefore we can assume that $n = 3$.

When the projection with center $P \in \mathbb{P}^3(\mathbb{C})$ is not birational from the curve C to its image $C' \subset \mathbb{P}^2$, we have $d' = \deg(C') \leq \frac{d}{2}$ hence $d' \leq d - 2$, and P is

the vertex of a cone K with base C' containing C . So we have to bound the number of such cones.

Let N be the dimension of the kernel of the restriction map

$$\alpha : H^0(\mathbb{P}^3, \mathcal{O}(d')) \rightarrow H^0(C, \mathcal{O}(d')).$$

Clearly N is bounded as a function of d and any $f \in \ker(\alpha)$ is an homogeneous polynomial of degree d' which vanishes on C .

Let $Z \subset \mathbb{P}^3(\mathbb{C}) \times \mathbb{P}^{N-1}(\mathbb{C})$ be the set of pairs (P, f) such that f is the equation of a cone K of vertex P . If $p_1 : \mathbb{P}^3 \times \mathbb{P}^{N-1} \rightarrow \mathbb{P}^3$ is the first projection, we have to bound the order of $p_1(Z)$. We note that this order is at most the number c of connected components of Z .

Now Z is defined by equations of bidegree $(\delta, 1)$, $\delta \leq d'$. Indeed f is homogeneous of degree d' and $(P, f) \in Z$ when all the derivatives of f , except those of order d' , vanish at P .

Let $L = \mathcal{O}(d', 1)$, $M = \dim H^0(\mathbb{P}^3 \times \mathbb{P}^N, L) - 1$, and

$$j : \mathbb{P}^3 \times \mathbb{P}^N \rightarrow \mathbb{P}^M$$

the Segre embedding. Since $j(Z)$ is the intersection of $j(\mathbb{P}^3 \times \mathbb{P}^N)$ with linear hyperplanes, Bézout theorem ([4], § 8.4) tells us that

$$c \leq \deg(j(\mathbb{P}^3 \times \mathbb{P}^N)).$$

Hence c is bounded by a function of d . □

Corollary. *Given any projective line $\Lambda \subset \mathbb{P}^n$, there exists a finite set Φ of order at most $A(d) + d$ in Λ such that, if $P \in \Lambda - \Phi$, the linear projection of center P maps C birationally onto its image.*

Proof. Since C is not equal to Λ , the cardinality of $C \cap \Lambda$ is at most d . So the Corollary follows from Theorem 1.

2 Successive minima

2.1

Let K be a number field, $[K : \mathbb{Q}]$ its degree over \mathbb{Q} , \mathcal{O}_K its ring of integers, $S = \text{Spec}(\mathcal{O}_K)$ the associated scheme and Σ the set of complex embeddings of K . Consider an hermitian vector bundle (E, h) over S , *i.e.* E is a torsion free \mathcal{O}_K -module of finite rank N and, for all $\sigma \in \Sigma$, the associated complex vector space $E_\sigma = E \otimes_{\mathcal{O}_K} \mathbb{C}$ is equipped with an hermitian scalar product h_σ . If $\bar{\sigma}$ is the conjugate of σ , we assume that the complex conjugation $E_\sigma \simeq E_{\bar{\sigma}}$ is an isometry.

If i is a positive integer, $i \leq N$, we let μ_i be the infimum of the set of real numbers r such that there exist $v_1, \dots, v_i \in E$, linearly independent over K ,

such that $\log \|v_\alpha\| \leq r$ for all $\alpha \leq i$. The number μ_i is thus the logarithm of the i -th successive minimum of (E, h) . Let

$$\mu = \frac{\mu_1 + \cdots + \mu_N}{N}. \quad (1)$$

2.2

If $E^\vee = \text{Hom}(E, \mathcal{O}_K)$ is the dual of E we let $\mathbb{P}(E^\vee)$ be the associated projective space, representing lines in E^\vee . Let $E_K^\vee = E^\vee \otimes_{\mathcal{O}_K} K$ and $X_K \subset \mathbb{P}(E_K^\vee)$ a smooth geometrically irreducible curve of genus g and degree d . We assume that the embedding of X_K into $\mathbb{P}(E_K^\vee)$ is defined by a complete linear series on X_K . We also assume that $g \geq 2$ and $d \geq 2g + 1$. The rank of E is thus $N = d + 1 - g$.

If X is the Zariski closure of X_K in $\mathbb{P}(E^\vee)$ and $\overline{\mathcal{O}(1)}$ the canonical hermitian line bundle on $\mathbb{P}(E^\vee)$, the Faltings height of X_K is the real number

$$h(X_K) = \widehat{\deg}(\hat{c}_1(\overline{\mathcal{O}(1)})^2 | X),$$

see [2] (3.1.1) and (3.1.5).

2.3

For any positive integer $i \leq N$ we define the integer f_i by the formulae

$$f_i = i - 1 \quad \text{if} \quad i - 1 \leq d - 2g$$

and

$$f_i = i - 1 + \alpha \quad \text{if} \quad i - 1 = d - 2g + \alpha, \quad 0 \leq \alpha \leq g.$$

Assume k and i are two positive integers, $k \leq N$, $i \leq N$. We let

$$h_{i,k} = \begin{cases} f_i & \text{if } i \leq k, \text{ } i = N - 1 \text{ or } i = N \\ f_k & \text{if } k \leq i \leq N - 2. \end{cases}$$

Finally, if $2 \leq k \leq N$, we let

$$B_k = \max_{i=2, \dots, N} \frac{h_{i,k}^2}{(i-1)h_{i,k} - \sum_{j=1}^{i-1} h_{j,k}}.$$

Theorem 2. *There exists a constant $C = C(d)$ such that, for every k such that $2 \leq k \leq N - 3$,*

$$B_k(\mu_{N+1-k} - \mu) + \frac{h(X_K)}{[K : \mathbb{Q}]} + 2d\mu \geq (2d - (N+1)B_k)(\mu - \mu_1) - C.$$

2.4

To prove Theorem 2 fix a positive integer $k \leq N - 3$ and choose elements x_1, \dots, x_N in E , linearly independent over K and such that

$$\log \|x_i\| = \mu_{N-i+1}, \quad 1 \leq i \leq N.$$

Fix integers n_α , $\alpha = k + 1, \dots, N - 2$, to be specified later (in § 2.6). If $1 \leq i \leq N$ we define

$$v_i = \begin{cases} x_i + n_i x_{i-1} & \text{if } k + 1 \leq i \leq N - 2 \\ x_i & \text{else.} \end{cases} \quad (2)$$

We get a complete flag $E_K = V_1 \supset V_2 \supset \dots \supset V_N$ by defining V_i to be the linear span of v_i, v_{i+1}, \dots, v_N .

When m is large enough the cup-product map

$$\varphi : E_K^{\otimes m} \rightarrow H^0(X_K, \mathcal{O}(m))$$

is surjective, hence $H^0(X_K, \mathcal{O}(m))$ is generated by the monomials

$$v_1^{\alpha_1} \dots v_N^{\alpha_N} = \varphi(v_1^{\otimes \alpha_1} \dots v_N^{\otimes \alpha_N}),$$

$\alpha_1 + \dots + \alpha_N = m$. A *special basis* of $H^0(X_K, \mathcal{O}(m))$ is a basis made of such monomials.

Let $r_1 \geq r_2 \geq \dots \geq r_N$ be N real numbers and $\mathbf{r} = (r_1, \dots, r_N)$. We define the weight of v_i to be r_i , the weight of a monomial in $E_K^{\otimes m}$ to be the sum of the weights of the v_i 's occuring in it, and the weight of a monomial $u \in H^0(X_K, \mathcal{O}(m))$ to be the minimum $wt_{\mathbf{r}}(u)$ of the weights of the monomials in the v_i 's mapping to u by φ . The weight $wt_{\mathbf{r}}(\mathcal{B})$ of a special basis \mathcal{B} is the sum of the weights of its elements, and $w_{\mathbf{r}}(m)$ is the minimum of the weight of a special basis of $H^0(X_K, \mathcal{O}(m))$.

When $r_1 \geq r_2 \geq \dots \geq r_N$ are natural integers there exists $e_{\mathbf{r}} \in \mathbb{N}$ such that, as m goes to infinity,

$$w_{\mathbf{r}}(m) = e_{\mathbf{r}} \frac{m^2}{2} + O(m)$$

([8], [7] Corollary 3.3).

Our next goal is to find an upper bound for $e_{\mathbf{r}}$.

2.5

For every positive integer $i \leq N$ we let e_i be the drop in degree of X_K when projected from $\mathbb{P}(E_K^\vee)$ to $\mathbb{P}(V_i^\vee)$. A criterion of Gieseker ([5], [7] Corollary 3.8) tells us that $e_{\mathbf{r}} \leq S$ with

$$S = \min_{1=i_0 < \dots < i_\ell = N} \sum_{j=0}^{\ell-1} (r_{i_j} - r_{i_{j+1}})(e_{i_j} + e_{i_{j+1}}).$$

Note that S is an increasing function in each variable e_i . Furthermore, it follows from Clifford's theorem and Riemann-Roch that

$$e_i \leq f_i \quad (3)$$

for every positive $i \leq N$ – see [7] proof of Theorem 4.4 (N.B.: in [7] Theorem 4.4 the filtration of V_0 has length $n + 1$, while $n = \dim V_0$. In our case, we start the filtration with V_1 , hence the discrepancy between our definition of f_i and [7] *loc. cit.*).

2.6

Let $w_1, \dots, w_N \in E_K^\vee$ be the dual basis of v_1, \dots, v_N . The linear projection from $\mathbb{P}(V_i^\vee)$ to $\mathbb{P}(V_{i+1}^\vee)$ has center the image \dot{w}_i of w_i .

If $y_1, \dots, y_N \in E_K^\vee$ is the dual basis of x_1, \dots, x_N , we get

$$w_i = \begin{cases} y_i + n_i z_i & \text{if } k \leq i \leq N - 3 \\ y_i & \text{else,} \end{cases}$$

where $z_i + y_{i+1}$ is a linear combination of y_{i+2}, y_{i+3}, \dots with coefficients depending only on n_{i+1}, n_{i+2}, \dots . When $n \neq m$ are two integers, the vectors $y_i + n z_i$ and $y_i + m z_i$ are linearly independent over K , therefore their images in $\mathbb{P}(V_i^\vee)$ are distinct. Since $e_{N-3} \leq f_{N-3}$ and $g \geq 2$ we get $e_{N-3} \leq d - 3$, therefore the image of X_K in $\mathbb{P}(V_i^\vee)$, $i \leq N - 3$, has degree at least 3. Furthermore $\dim \mathbb{P}(V_i^\vee) \geq 3$. By Theorem 1 and its Corollary, it follows that we can choose n_i such that $0 \leq n_i < A(d) + d$ and the projection of $\mathbb{P}(V_i^\vee)$ to $\mathbb{P}(V_{i+1}^\vee)$ does not change the degree of the image of X_K . We fix the integers n_i , $k \leq i \leq N - 3$, with this property. Hence we have

$$e_i = e_k \quad \text{whenever} \quad k \leq i \leq N - 2. \quad (4)$$

2.7

From (3) and (4) we conclude that

$$e_i \leq h_{i,k} \quad \text{if} \quad 1 \leq i \leq N$$

(see 2.3). Hence, by Morrison's main combinatorial theorem, [7] Corollary 4.3, for any decreasing sequence of real numbers $r_1 \geq r_2 \geq \dots \geq r_N$ we have, if $k \geq 2$,

$$S \leq \psi(\mathbf{r})$$

with

$$\psi(\mathbf{r}) = B_k \cdot \sum_{j=1}^N (r_j - r_N).$$

So, when $r_1 \geq r_2 \geq \dots \geq r_N = 0$ is a decreasing sequence of real numbers,

$$e_{\mathbf{r}} \leq \psi(\mathbf{r}).$$

From the proof of Theorem 1 in [9] we deduce that, letting

$$s_i = \log \|v_i\| - \log \|v_N\|, \quad 1 \leq i \leq N,$$

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 2d \log \|v_N\| + \psi(s_1, s_2, \dots, s_{N-1}, 0) \geq 0. \quad (5)$$

From (2) above we get

$$\log \|v_i\| \leq \log \|x_{i-1}\| + \log(1 + n_i) \quad \text{if } k+1 \leq i \leq N-2$$

and $\log \|v_i\| = \log \|x_i\|$ otherwise.

Since $\log \|x_i\| = \mu_{N+1-i}$ and $n_i < A + d$ we deduce that

$$\psi(s_1, s_2, \dots, s_{N-1}, 0) \leq B_k \left(\sum_{i=1}^N (\mu_i - \mu_1) + \mu_{N+1-k} - \mu_3 + (N-2-k) \log(A+d) \right). \quad (6)$$

From (1), (5) and (6) it follows that

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 2d \mu_1 + B_k(N(\mu - \mu_1) + \mu_{N+1-k} - \mu_3) + C \geq 0 \quad (7)$$

for some constant $C = C(d)$. Since $\mu_3 \geq \mu_1$ the inequality in Theorem 2 follows from (7).

References

- [1] Arbarello, E.; Cornalba, M.; Griffiths, P.A.; Harris, J. Geometry of algebraic curves. Volume I. Grundlehren der mathematischen Wissenschaften, 267. New York etc.: Springer-Verlag (1985).
- [2] Bost, J.-B.; Gillet, H.; Soulé, C. Heights of projective varieties and positive Green forms. J. Am. Math. Soc. 7, No.4, 903-1027 (1994).
- [3] Calabri, A.; Ciliberto, C. : On special projections of varieties: epitome to a theorem of Beniamino Segre. Adv. Geom. 1 , no. 1, 97–106 (2001).
- [4] Fulton, W. Intersection theory. 2nd ed. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 2. Berlin: Springer-Verlag (1998).
- [5] Gieseker, D. Global moduli for surfaces of general type. Invent. Math. 43, 233-282 (1977).
- [6] Hartshorne, R. Algebraic geometry. Corr. 3rd printing. Graduate Texts in Mathematics, 52. New York-Heidelberg-Berlin: Springer- Verlag (1983).
- [7] Morrison, I. Projective stability of ruled surfaces. Invent. Math. 56, 269-304 (1980).
- [8] Mumford, D. Stability of projective varieties. Enseign. Math., II. Sr. 23, 39-110 (1977).
- [9] Soulé, C. Successive minima on arithmetic varieties. Compos. Math. 96, No.1, 85-98 (1995).

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